

FRW cosmological models with integrable and nonintegrable differential equations of state

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Abstract

The basic models of modern cosmology work in FRW spacetime. Hence follows that it is important to study the physical and mathematical nature of FRW cosmological models. In this work, we consider FRW models with the differential equations of state. Some new classes integrable and nonintegrable FRW cosmological models were constructed. It is remarkable that all proposed integrable and nonintegrable FRW models admit exact solutions. For some of them, such exact solutions are presented. Some artificial two-dimensional FRW models were also proposed. Finally in Appendix, we extend the obtained results for g-essence models and for its two reductions: k-essence and f-essence.

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1 Introduction

FRW models play a central role in modern cosmology. In particular, almost all popular theoretical models of dark energy work in FRW spacetime. Again almost all models of dark energy meet some difficulties like cosmological constant problems, fine-tuning problems and so on. One of consequences of such difficulties of modern cosmology is the necessity more carefully investigate the basics of General Relativity (GR), in particular, FRW cosmology. One of the poorly studied sector of modern theoretical cosmology is the problem of integrability of cosmological models, especially, FRW cosmological models. In our previous papers [1]-[3], we have studied the relationship between the basic cosmological equations, namely, the Friedmann equations

$$\begin{aligned} p &= -2\dot{H} - 3H^2, \\ \rho &= 3H^2 \end{aligned} \tag{1.1}$$

and some important equations known from the other branches of physics and/or mathematics. These last equations include:

a) The Ramanujan equation

$$\dot{F} = \frac{F^2 - E}{12}, \tag{1.3}$$

$$\dot{E} = \frac{FE - J}{3}, \tag{1.4}$$

$$\dot{J} = \frac{FJ - E^2}{2}. \tag{1.5}$$

b) The Chazy-III equation

$$\ddot{y} = 2y\ddot{y} - 3\dot{y}^2. \tag{1.6}$$

c) The Lorenz oscillator equation

$$X_N = \sigma(Y - X), \tag{1.7}$$

$$Y_N = X(\delta - Z) - Y, \tag{1.8}$$

$$Z_N = XY - \beta Z. \tag{1.9}$$

d) Painlevé equations. These six equations, traditionally called Painlevé I-VI, are as follows:

$P_I - \text{equation}$	$\ddot{y} = 6y^2 + t$	(1.10)
$P_{II} - \text{equation}$	$\ddot{y} = 2y^3 + ty + \alpha$	
$P_{III} - \text{equation}$	$\ddot{y} = \frac{1}{y}\dot{y}^2 - \frac{1}{t}(\dot{y} - \alpha y^2 - \beta) + \gamma y^3 + \frac{\delta}{y}$	
$P_{IV} - \text{equation}$	$\ddot{y} = \frac{1}{2y}\dot{y}^2 + 1.5y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\delta}{y}$	
$P_V - \text{equation}$	$\ddot{y} = (\frac{1}{2y} + \frac{1}{y-1})\dot{y}^2 - \frac{1}{t}(\dot{y} - \gamma y) + \frac{(y-1)^2}{t^2}(\alpha y + \frac{\beta}{y}) + \frac{\delta y(y+1)}{y-1}$	
$P_{VI} - \text{equation}$	$\ddot{y} = \varphi(t)\dot{y}^2 - \xi(t)\dot{y} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}[\alpha + \frac{\beta t}{y^2} + \zeta(t)]$	

Here $\varphi(z) = 0.5[y^{-1} + (y-1)^{-1} + (y-z)^{-1}]$, $\xi(z) = [z^{-1} + (z-1)^{-1} + (y-z)^{-1}]$, $\zeta(z) = [\gamma(z-1)(y-1)^{-2} + \delta z(z-1)(y-z)^{-2}]$.

In this work, we study the FRW cosmological models with the differential equations of state (EoS). To construct integrable and nonintegrable reductions of such FRW cosmological models we use the implantation method. In particular, to construct integrable FRW models in one dimensions we use six Painlevé equations (see also [1]-[3]).

The paper is organized as follows. In section 2, we present the basic equations of FRW cosmology. Section 3 is devoted to study the FRW models with the half EoS and Section 4 to the FRW models with the full EoS. In the next section 5, we present the so-called D - models. Some artificial two-dimensional models were constructed in the section 6. The last section 7 is devoted to the conclusion.

2 Basic gravitational equations

We start from the classical GR case. In this GR case, the standard gravitational action has the form

$$S = \int \sqrt{-g} d^4x (R + L_m - \Lambda), \quad (2.1)$$

where R is the scalar curvature and L_m is the Lagrangian of the matter. We work with the FRW spacetime which has the metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right], \quad (2.2)$$

where $a(t)$ is the scale factor, $k = -1, 0, 1$ represent the three-dimensional space with the negative, zero, and positive spatial curvature, respectively. In this case the Ricci scalar reads as

$$R = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right], \quad (2.3)$$

where a dot represents differentiation with respect to t . The Friedmann equations read as

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{\Lambda}{3} - \frac{k}{a^2} + \frac{8\pi G}{3}\rho, \quad \frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p), \quad (2.4)$$

where we note that $p < -\rho/3$ implies repulsive gravitation if $\Lambda = 0$. Recaling the Hubble parameter $H = \dot{a}a^{-1}$ these equations can be written as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \quad \dot{H} = -4\pi G(\rho + p) + \frac{k}{a^2}. \quad (2.5)$$

If the FRW spacetime is filled with a fluid of energy density ρ and pressure p , then the conservation law could be derived from the Friedmann equations (2.5) as

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (2.6)$$

In this work, we consider the case: $k = \Lambda = 0$ and set $8\pi G = 1$. So finally the equations for the action (2.1) we can write in *the H-form*

$$p = -2\dot{H} - 3H^2, \quad (2.7)$$

$$\rho = 3H^2, \quad (2.8)$$

$$\dot{\rho} = -3H(\rho + p), \quad (2.9)$$

in *the N-form*

$$p = -2\ddot{N} - 3\dot{N}^2, \quad (2.10)$$

$$\rho = 3\dot{N}^2, \quad (2.11)$$

$$\dot{\rho} = -3\dot{N}(\rho + p) \quad (2.12)$$

or in *the a-form*

$$p = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}, \quad (2.13)$$

$$\rho = 3\frac{\dot{a}^2}{a^2}, \quad (2.14)$$

$$\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + p), \quad (2.15)$$

where $N = \ln a$. Note that one equation among three is redundant. Here we may choose the first two Friedmann equations as two relevant equations. To find H we note that the general solution of the equation (2.7) we can write as

$$H(a) = \sqrt{-a^{-3} \int p a^2 da}. \quad (2.16)$$

This formula tells us that the EoS has the form

$$\rho = -3a^{-3} \int p a^2 da \quad (2.17)$$

or

$$p = -\rho - 3^{-1} a \rho_a \quad (2.18)$$

so that its parameter takes the form

$$\omega = -1 - 3^{-1} a (\ln \rho)_a. \quad (2.19)$$

3 Differential equations of state: 0.5 - models

3.1 Integrable 0.5 - models

In this subsection we consider some integrable 0.5 - models that means models with the half EoS ($half \equiv 0.5$). As the integrable cells (which we implant to the body of the original gravitational system of equations), we use Painlevé equations. The function of these integrable cells are to convert the original gravitational (nonintegrable) system into the integrable system.

3.1.1 ρ - models

Let's go further or, if exactly, go back. To the Friedmann equations e.g. (2.7)-(2.9) and consider the following its extension

$$p = -2\dot{H} - 3H^2, \quad (3.1)$$

$$\rho = 3H^2, \quad (3.2)$$

$$\ddot{\rho} = \rho^{-1}\dot{\rho}^2 - t^{-1}(\dot{\rho} - \alpha\rho^2 - \beta) + \gamma\rho^3 + \delta\rho^{-1}, \quad (3.3)$$

$$\dot{\rho} = -3H(\rho + p), \quad (3.4)$$

where $\alpha, \beta, \gamma, \delta$ are consts. It is the $A_{III A}$ - model (see below). We guess that this system is integrable due to integrability of the equation (3.3) which is nothing but the P_{III} - equation and which plays the role of the integrable cell (see e.g. [2]-[4]). To solve it, we start from the equation (3.3). In particular, it has the following well-known particular solutions [4]:

$$\rho(t) \equiv \rho(t; \mu, -\mu\kappa^2, \lambda, -\lambda\kappa^4) = \kappa, \quad (3.5)$$

$$\rho(t) \equiv \rho(t; 0, -\mu, 0, \mu\kappa) = \kappa t, \quad (3.6)$$

$$\rho(t) \equiv \rho(t; 2\kappa + 3, -2\kappa + 1, 1, -1) = \frac{t + \kappa}{t + \kappa + 1}, \quad (3.7)$$

$$\rho(t) \equiv \rho(t; \mu, 0, 0, -\mu\kappa^3) = \kappa\sqrt[3]{t}, \quad (3.8)$$

$$\rho(t) \equiv \rho(t; 0, -2\kappa, 0, 4\kappa\mu - \lambda^2) = t[\kappa(\ln t)^2 + \lambda \ln t + \mu], \quad (3.9)$$

$$\rho(t) \equiv \rho(t; -\nu^2\lambda, 0, \nu^2(\lambda^2 - 4\kappa\mu), 0) = \frac{t^{\nu-1}}{\kappa t^{2\nu} + \lambda t^\nu + \mu}, \quad (3.10)$$

$$\rho(t) \equiv \rho(t; 0.5\epsilon) = -\epsilon_1(\ln \phi)_t \quad (3.11)$$

and so on. Here

$$\phi(t) = t^\nu[C_1 J_\nu(\zeta) + C_2 Y_\nu(\zeta)], \quad (3.12)$$

where $C_i = \text{consts}$, $\zeta = \sqrt{\epsilon_1\epsilon_2}t$, $\nu = 0.5\alpha\epsilon_1$ and $J_\nu(\zeta), Y_\nu(\zeta)$ are Bessel functions. We are here modest and work just with the simplest solutions.

i) First let's consider the solution (3.5):

$$\rho(t) = \kappa = \text{const}. \quad (3.13)$$

From (3.2) we get that

$$H(t) = H_0 = 3^{-0.5}\kappa^{0.5} = \text{const}, \quad (3.14)$$

so that this solution corresponds to the de Sitter case.

ii) Next, let's consider the other simplest solution namely the solution (3.6):

$$\rho(t) = \kappa t, \quad (3.15)$$

where we assume that $\kappa > 0$. Then the equation (3.2) gives

$$H = \pm\sqrt{\frac{\kappa}{3}}t^{0.5} \quad (3.16)$$

so that we have

$$a = a_0 e^{\pm\sqrt{\frac{4\kappa}{27}}t^{1.5}}. \quad (3.17)$$

For this particular solution the EoF has the form

$$p = -\rho \mp 3^{-0.5}\kappa^{0.75}\rho^{-0.25}. \quad (3.18)$$

The corresponding EoF parameter reads as

$$\omega = -1 \mp 3^{-0.5}\kappa^{-0.5}t^{-1.5}. \quad (3.19)$$

For this case we have

$$\dot{a} = \pm a_0 \sqrt{\frac{\kappa}{3}}t^{0.5} e^{\pm\sqrt{\frac{4\kappa}{27}}t^{1.5}} \quad (3.20)$$

and

$$\ddot{a} = \pm \frac{\kappa a_0}{3} t^{-0.5} \left[\sqrt{\frac{3}{4\kappa}} \pm t^{1.5} \right] e^{\pm \sqrt{\frac{4\kappa}{27}} t^{1.5}}. \quad (3.21)$$

Let's calculate the deceleration parameter. We have

$$q = -t^{-1.5} \left[\sqrt{\frac{3}{4\kappa}} \pm t^{1.5} \right]. \quad (3.22)$$

Hence we see that $q = 0$ as $t_0 = \sqrt[3]{\frac{3}{4\kappa}}$. It means that this solution describes the deceleration and acceleration phases of the expansion of the universe and $t_0 = \sqrt[3]{\frac{3}{4\kappa}}$ is the transition point. Similarly, we can explore the other solutions of the A_{IIIA} - model (3.1) - (3.4) as well as the other ρ - models. Now we present the list of ρ - models.

1) **A_I - models.**

$A_{IA} - model$	$\ddot{\rho} = 6\rho^2 + t$	(3.23)
$A_{IB} - model$	$\rho_{aa} = 6\rho^2 + a$	
$A_{IC} - model$	$\rho_{NN} = 6\rho^2 + N$	
$A_{ID} - model$	$\rho_{HH} = 6\rho^2 + H$	

2) **A_{II} - models.**

$A_{IIA} - model$	$\ddot{\rho} = 2\rho^3 + t\rho + \alpha$	(3.24)
$A_{IIB} - model$	$\rho_{aa} = 2\rho^3 + a\rho + \alpha$	
$A_{IIC} - model$	$\rho_{NN} = 2\rho^3 + N\rho + \alpha$	
$A_{IID} - model$	$\rho_{HH} = 2\rho^3 + H\rho + \alpha$	

3) **A_{III} - models.**

$A_{IIIA} - model$	$\ddot{\rho} = \rho^{-1}\rho^2 - t^{-1}(\dot{\rho} - \alpha\rho^2 - \beta) + \gamma\rho^3 + \delta\rho^{-1}$	(3.25)
$A_{IIIB} - model$	$\rho_{aa} = \rho^{-1}\rho_a^2 - a^{-1}(\rho_a - \alpha\rho^2 - \beta) + \gamma\rho^3 + \delta\rho^{-1}$	
$A_{IIIC} - model$	$\rho_{NN} = \rho^{-1}\rho_N^2 - N^{-1}(\rho_N - \alpha\rho^2 - \beta) + \gamma\rho^3 + \delta\rho^{-1}$	
$A_{IIID} - model$	$\rho_{HH} = \rho^{-1}\rho_H^2 - H^{-1}(\rho_H - \alpha\rho^2 - \beta) + \gamma\rho^3 + \delta\rho^{-1}$	

4) **A_{IV} - models.**

$A_{IVA} - model$	$\ddot{\rho} = 0.5\rho^{-1}\rho^2 + 1.5\rho^3 + 4t\rho^2 + 2(t^2 - \alpha)\rho + \beta\rho^{-1}$	(3.26)
$A_{IVB} - model$	$\rho_{aa} = 0.5\rho^{-1}\rho_a^2 + 1.5\rho^3 + 4a\rho^2 + 2(a^2 - \alpha)\rho + \beta\rho^{-1}$	
$A_{IVC} - model$	$\rho_{NN} = 0.5\rho^{-1}\rho_N^2 + 1.5\rho^3 + 4N\rho^2 + 2(N^2 - \alpha)\rho + \beta\rho^{-1}$	
$A_{IVD} - model$	$\rho_{HH} = 0.5\rho^{-1}\rho_H^2 + 1.5\rho^3 + 4H\rho^2 + 2(H^2 - \alpha)\rho + \beta\rho^{-1}$	

5) A_V - **models.**

$A_{VA} - model$	$\ddot{\rho} = \phi\dot{\rho}^2 - t^{-1}(\dot{\rho} - \gamma\rho) + t^{-2}(\rho - 1)^2(\alpha\rho + \beta\rho^{-1}) + \delta\psi$	(3.27)
$A_{VB} - model$	$\rho_{aa} = \phi\rho_a^2 - a^{-1}(\rho_a - \gamma\rho) + a^{-2}(\rho - 1)^2(\alpha\rho + \beta\rho^{-1}) + \delta\psi$	
$A_{VC} - model$	$\rho_{NN} = \phi\rho_N^2 - N^{-1}(\rho_N - \gamma\rho) + N^{-2}(\rho - 1)^2(\alpha\rho + \beta\rho^{-1}) + \delta\psi$	
$A_{VD} - model$	$\rho_{HH} = \phi\rho_H^2 - H^{-1}(\rho_H - \gamma\rho) + H^{-2}(\rho - 1)^2(\alpha\rho + \beta\rho^{-1}) + \delta\psi$	

Here $\phi = 0.5\rho^{-1} + (\rho - 1)^{-1}$, $\psi = \rho(\rho + 1)(\rho - 1)^{-1}$.

6) A_{VI} - **models.**

$A_{VIA} - model$	$\ddot{\rho} = \varphi(t)\dot{\rho}^2 - \xi(t)\dot{\rho} - \frac{\rho(\rho-1)(\rho-t)}{t^2(t-1)^2}[\alpha + \frac{\beta t}{\rho^2} + \zeta(t)]$	(3.28)
$A_{VIB} - model$	$\rho_{aa} = \varphi(a)\rho_a^2 - \xi(a)\rho_a - \frac{\rho(\rho-1)(\rho-a)}{a^2(a-1)^2}[\alpha + \frac{\beta a}{\rho^2} + \zeta(a)]$	
$A_{VIC} - model$	$\rho_{NN} = \varphi(N)\rho_N^2 - \xi(N)\rho_N - \frac{\rho(\rho-1)(\rho-N)}{N^2(N-1)^2}[\alpha + \frac{\beta N}{\rho^2} + \zeta(N)]$	
$A_{VID} - model$	$\rho_{HH} = \varphi(H)\rho_H^2 - \xi(H)\rho_H - \frac{\rho(\rho-1)(\rho-H)}{H^2(H-1)^2}[\alpha + \frac{\beta H}{\rho^2} + \zeta(H)]$	

Here $\varphi(z) = 0.5[\rho^{-1} + (\rho - 1)^{-1} + (\rho - z)^{-1}]$, $\xi(z) = [z^{-1} + (z - 1)^{-1} + (\rho - z)^{-1}]$, $\zeta(z) = [\gamma(z - 1)(\rho - 1)^{-2} + \delta z(z - 1)(\rho - z)^{-2}]$.

3.1.2 p - models

Now we want consider one of important part of integrable 0.5 models namely p - models. Let's implant the integrable cell, e.g. the B_{IIIB} - model (see below), to the body of the gravitational equations. The aim of this implantation is the original (in general nonintegrable) system convert into the integrable system. If this integrable cell is, e.g. the B_{IIIB} - model, as result we get the following closed system of the equations

$$p = -2\dot{H} - 3H^2, \quad (3.29)$$

$$\rho = 3H^2, \quad (3.30)$$

$$p_{aa} = p^{-1}p_a^2 - a^{-1}(p_a - \alpha p^2 - \beta) + \gamma p^3 + \delta p^{-1}, \quad (3.31)$$

$$\dot{\rho} = -3H(\rho + p), \quad (3.32)$$

where $\alpha, \beta, \gamma, \delta$ are const. This system we also call the B_{IIIB} - model, which is (as we expect and believe) integrable. Also we note that the equation (3.31) is the P_{III} - equation and plays the role of the integrable cell. As an integrable, the system (3.29)-(3.32) admits (may be infinite number) exact solutions. The construction its exact solutions we start from the equation (3.31).

In particular, the equation (3.31) has the following particular solutions (see e.g. [4]):

$$p(a) \equiv p(a; \mu, -\mu\kappa^2, \lambda, -\lambda\kappa^4) = \kappa, \quad (3.33)$$

$$p(a) \equiv p(a; 0, -\mu, 0, \mu\kappa) = \kappa a, \quad (3.34)$$

$$p(a) \equiv p(a; 2\kappa + 3, -2\kappa + 1, 1, -1) = \frac{a + \kappa}{a + \kappa + 1}, \quad (3.35)$$

$$p(a) \equiv p(a; \mu, 0, 0, -\mu\kappa^3) = \kappa \sqrt[3]{a}, \quad (3.36)$$

$$p(a) \equiv p(a; 0, -2\kappa, 0, 4\kappa\mu - \lambda^2) = a[\kappa(\ln a)^2 + \lambda \ln a + \mu], \quad (3.37)$$

$$p(a) \equiv p(a; -\nu^2\lambda, 0, \nu^2(\lambda^2 - 4\kappa\mu), 0) = \frac{a^{\nu-1}}{\kappa a^{2\nu} + \lambda a^\nu + \mu}, \quad (3.38)$$

$$p(a) \equiv p(a; 0.5\epsilon) = -\epsilon_1(\ln \phi)_a \quad (3.39)$$

and so on. Here

$$\phi(a) = a^\nu [C_1 J_\nu(\zeta) + C_2 Y_\nu(\zeta)], \quad (3.40)$$

where $C_i = \text{consts}$, $\zeta = \sqrt{\epsilon_1 \epsilon_2} a$, $\nu = 0.5\alpha\epsilon_1$ and $J_\nu(\zeta), Y_\nu(\zeta)$ are Bessel functions. Now we consider some simplest solutions.

i) Let's start from the solution (3.33)

$$p(a) = \kappa = \text{const}. \quad (3.41)$$

To find H we use the formula (2.16). As result we get

$$H(a) = \sqrt{-a^{-3}(3^{-1}\kappa a^3 + C)} = \sqrt{-3^{-1}\kappa - C a^{-3}} = \sqrt{3^{-1}\rho_0 a^{-3} + 3^{-1}\Lambda}. \quad (3.42)$$

where $\rho_0 = -3C$, $\Lambda = -\kappa$. It is nothing but Λ CDM model. So this solution corresponds to the Λ CDM cosmology. In this case, the above formulas give

$$p = -\Lambda, \quad \rho = \rho_0 a^{-3} + \Lambda \quad (3.43)$$

which corresponds the EoF parameter

$$\omega = -1 + \rho_0(\rho_0 + \Lambda a^3)^{-1}. \quad (3.44)$$

ii) Now let's we consider the next simplest solution namely the solution (3.34):

$$p(a) = \kappa a. \quad (3.45)$$

Eq. (2.16) gives

$$H(a) = \sqrt{-a^{-3}(0.25\kappa a^4 + C)}. \quad (3.46)$$

The corresponding energy density is given by

$$\rho = -3a^{-3}(0.25\kappa a^4 + C). \quad (3.47)$$

For this particular solution the EoF and its parameter take the form

$$\rho = -0.75p - 3C\kappa^3 p^{-3} \quad (3.48)$$

and

$$\omega = -\frac{\kappa a^4}{0.75\kappa a^4 + 3C}, \quad (3.49)$$

respectively. In the limit $a \rightarrow \infty$ we get $\omega \rightarrow -4/3$.

Similarly we can write B_J - models induced by the other P_J - equations. Here the list of such models.

1) **B_I - models.**

$B_{IA} - \text{model}$	$\ddot{p} = 6p^2 + t$
$B_{IB} - \text{model}$	$p_{aa} = 6p^2 + a$
$B_{IC} - \text{model}$	$p_{NN} = 6p^2 + N$
$B_{ID} - \text{model}$	$p_{HH} = 6p^2 + H$

(3.50)

2) **B_{II} - models.**

$B_{IIA} - model$	$\ddot{p} = 2p^3 + tp + \alpha$
$B_{IIB} - model$	$p_{aa} = 2p^3 + ap + \alpha$
$B_{IIC} - model$	$p_{NN} = 2p^3 + Np + \alpha$
$B_{IID} - model$	$p_{HH} = 2p^3 + Hp + \alpha$

(3.51)

3) **B_{III} - models.**

$B_{IIIA} - model$	$\ddot{p} = p^{-1}\dot{p}^2 - t^{-1}(\dot{p} - \alpha p^2 - \beta) + \gamma p^3 + \delta p^{-1}$
$B_{IIIB} - model$	$p_{aa} = p^{-1}p_a^2 - a^{-1}(p_a - \alpha p^2 - \beta) + \gamma p^3 + \delta p^{-1}$
$B_{IIIC} - model$	$p_{NN} = p^{-1}p_N^2 - N^{-1}(p_N - \alpha p^2 - \beta) + \gamma p^3 + \delta p^{-1}$
$B_{IIID} - model$	$p_{HH} = p^{-1}p_H^2 - H^{-1}(p_H - \alpha p^2 - \beta) + \gamma p^3 + \delta p^{-1}$

(3.52)

4) **B_{IV} - models.**

$B_{IVA} - model$	$\ddot{p} = 0.5p^{-1}\dot{p}^2 + 1.5p^3 + 4tp^2 + 2(t^2 - \alpha)p + \beta p^{-1}$
$B_{IVB} - model$	$p_{aa} = 0.5p^{-1}p_a^2 + 1.5p^3 + 4ap^2 + 2(a^2 - \alpha)p + \beta p^{-1}$
$B_{IVC} - model$	$p_{NN} = 0.5p^{-1}p_N^2 + 1.5p^3 + 4Np^2 + 2(N^2 - \alpha)p + \beta p^{-1}$
$B_{IVD} - model$	$p_{HH} = 0.5p^{-1}p_H^2 + 1.5p^3 + 4Hp^2 + 2(H^2 - \alpha)p + \beta p^{-1}$

(3.53)

5) **B_V - models.**

$B_{VA} - model$	$\ddot{p} = \phi \dot{p}^2 - t^{-1}(\dot{p} - \gamma p) + t^{-2}(p - 1)^2(\alpha p + \beta p^{-1}) + \delta \psi$
$B_{VB} - model$	$p_{aa} = \phi p_a^2 - a^{-1}(p_a - \gamma p) + a^{-2}(p - 1)^2(\alpha p + \beta p^{-1}) + \delta \psi$
$B_{VC} - model$	$p_{NN} = \phi p_N^2 - N^{-1}(p_N - \gamma p) + N^{-2}(p - 1)^2(\alpha p + \beta p^{-1}) + \delta \psi$
$B_{VD} - model$	$p_{HH} = \phi p_H^2 - H^{-1}(p_H - \gamma p) + H^{-2}(p - 1)^2(\alpha p + \beta p^{-1}) + \delta \psi$

(3.54)

Here $\phi = 0.5p^{-1} + (p - 1)^{-1}$, $\psi = p(p + 1)(p - 1)^{-1}$.

6) **B_{VI} - models.**

$B_{VIA} - model$	$\ddot{p} = \varphi(t)p^2 - \xi(t)\dot{p} - \frac{p(p-1)(p-t)}{t^2(t-1)^2}[\alpha + \frac{\beta t}{p^2} + \zeta(t)]$
$B_{VIB} - model$	$p_{aa} = \varphi(a)p_a^2 - \xi(a)p_a - \frac{p(p-1)(p-a)}{a^2(a-1)^2}[\alpha + \frac{\beta a}{p^2} + \zeta(a)]$
$B_{VIC} - model$	$p_{NN} = \varphi(N)p_N^2 - \xi(N)p_N - \frac{p(p-1)(p-N)}{N^2(N-1)^2}[\alpha + \frac{\beta N}{p^2} + \zeta(N)]$
$B_{VID} - model$	$p_{HH} = \varphi(H)p_H^2 - \xi(H)p_H - \frac{p(p-1)(p-H)}{H^2(H-1)^2}[\alpha + \frac{\beta H}{p^2} + \zeta(H)]$

(3.55)

Here $\varphi(z) = 0.5[p^{-1} + (p-1)^{-1} + (p-z)^{-1}]$, $\xi(z) = [z^{-1} + (z-1)^{-1} + (p-z)^{-1}]$, $\zeta(z) = [\gamma(z-1)(p-1)^{-2} + \delta z(z-1)(p-z)^{-2}]$.

3.2 Nonintegrable 0.5 - models

Now we give some examples nonintegrable 0.5 - models.

3.2.1 ρ - models

1) Λ cosmology.

i) Example 1:

$$\ddot{\rho} = 0.5\Lambda - 1.5\rho^2. \quad (3.56)$$

ii) Example 2:

$$\rho_{aa} = 0.5\Lambda - 1.5\rho_a^2. \quad (3.57)$$

iii) Example 3:

$$\rho_{HH} = 0.5\Lambda - 1.5\rho_H^2. \quad (3.58)$$

iv) Example 4:

$$\rho_{NN} = 0.5\Lambda - 1.5\rho_N^2. \quad (3.59)$$

2) **Pinney cosmology.** It induced by the Pinney equation. Let us present 4 examples of such models.

i) Example 1:

$$\ddot{\rho} = \xi(t)\rho + \frac{k}{\rho^3}. \quad (3.60)$$

ii) Example 2:

$$\rho_{aa} = \xi(a)\rho + \frac{k}{\rho^3}. \quad (3.61)$$

iii) Example 3:

$$\rho_{HH} = \xi(H)\rho + \frac{k}{\rho^3}. \quad (3.62)$$

iv) Example 4:

$$\rho_{NN} = \xi(N)\rho + \frac{k}{\rho^3}. \quad (3.63)$$

3) **Schrödinger cosmology.** For this model we give 4 submodels.

i) Example 1:

$$\ddot{\rho} = u(t)\rho + k\rho, \quad (3.64)$$

ii) Example 2:

$$\rho_{aa} = u(a)\rho + k\rho. \quad (3.65)$$

iii) Example 3:

$$\rho_{HH} = u(H)\rho + k\rho. \quad (3.66)$$

iv) Example 4:

$$\rho_{NN} = u(N)\rho + k\rho. \quad (3.67)$$

4) **Hypergeometric cosmology.** Let us we present 4 examples.

i) Example 1:

$$\ddot{\rho} = t^{-1}(1-t)^{-1}\{[(\alpha + \beta + 1)t - \gamma]\dot{\rho} + \alpha\beta\rho\}. \quad (3.68)$$

ii) Example 2:

$$\rho_{aa} = a^{-1}(1-a)^{-1}\{[(\alpha + \beta + 1)a - \gamma]\rho_a + \alpha\beta\rho\}. \quad (3.69)$$

iii) Example 3:

$$\rho_{HH} = H^{-1}(1-H)^{-1}\{[(\alpha + \beta + 1)H - \gamma]\rho_H + \alpha\beta\rho\}. \quad (3.70)$$

iv) Example 4:

$$\rho_{NN} = N^{-1}(1-N)^{-1}\{[(\alpha + \beta + 1)N - \gamma]\rho_N + \alpha\beta\rho\}. \quad (3.71)$$

3.2.2 p - models

1) **Λ cosmology.**

i) Example 1:

$$\ddot{p} = 0.5\Lambda - 1.5p^2. \quad (3.72)$$

ii) Example 2:

$$p_{aa} = 0.5\Lambda - 1.5p_a^2. \quad (3.73)$$

iii) Example 3:

$$p_{HH} = 0.5\Lambda - 1.5p_H^2. \quad (3.74)$$

iv) Example 4:

$$p_{NN} = 0.5\Lambda - 1.5p_N^2. \quad (3.75)$$

2) **Pinney cosmology.** It induced by the Pinney equation. Let us present 4 examples of such models.

i) Example 1:

$$\ddot{p} = \xi(t)p + \frac{k}{p^3}, \quad (3.76)$$

where $\xi = \xi(t)$, $k = \text{const.}$

ii) Example 2:

$$p_{aa} = \xi(a)p + \frac{k}{p^3}. \quad (3.77)$$

iii) Example 3:

$$p_{HH} = \xi(H)p + \frac{k}{p^3}. \quad (3.78)$$

iv) Example 4:

$$p_{NN} = \xi(N)p + \frac{k}{p^3}. \quad (3.79)$$

3) **Schrödinger cosmology.** For this model also write 5 submodels.

i) Example 1:

$$\ddot{p} = u(t)p + kp, \quad (3.80)$$

where $u = u(t)$, $k = \text{const.}$

ii) Example 2:

$$p_{aa} = u(a)p + kp. \quad (3.81)$$

iii) Example 3:

$$p_{HH} = u(H)p + kp. \quad (3.82)$$

iv) Example 4:

$$p_{NN} = u(N)p + kp. \quad (3.83)$$

4) **Hypergeometric cosmology.** Let us we present 4 examples.

i) Example 1:

$$\ddot{p} = t^{-1}(1-t)^{-1}\{[(\alpha + \beta + 1)t - \gamma]\dot{p} + \alpha\beta p\}. \quad (3.84)$$

ii) Example 2:

$$p_{aa} = a^{-1}(1-a)^{-1}\{[(\alpha + \beta + 1)a - \gamma]p_a + \alpha\beta p\}. \quad (3.85)$$

iii) Example 3:

$$p_{HH} = H^{-1}(1-H)^{-1}\{[(\alpha + \beta + 1)H - \gamma]p_H + \alpha\beta p\}. \quad (3.86)$$

iv) Example 4:

$$p_{NN} = N^{-1}(1-N)^{-1}\{[(\alpha + \beta + 1)N - \gamma]p_N + \alpha\beta p\}. \quad (3.87)$$

4 Differential equations of state: 1.0 - models

Here we present some examples integrable and nonintegrable 1.0 - models that means models with the full EoS ($full \equiv 1.0$).

4.1 Integrable 1.0 - models

Here the list integrable 1.0 - models.

$K_I - model$	$p_{\rho\rho} = 6p^2 + \rho$	(4.1)
$K_{II} - model$	$p_{\rho\rho} = 2p^3 + \rho p + \alpha$	
$K_{III} - model$	$p_{\rho\rho} = \frac{1}{p}p_\rho^2 - \frac{1}{\rho}(p_\rho - \alpha p^2 - \beta) + \gamma p^3 + \frac{\delta}{p}$	
$K_{IV} - model$	$p_{\rho\rho} = \frac{1}{2p}p_\rho^2 + 1.5p^3 + 4\rho p^2 + 2(\rho^2 - \alpha)p + \frac{\delta}{p}$	
$K_V - model$	$p_{\rho\rho} = (\frac{1}{2p} + \frac{1}{p-1})p_\rho^2 - \frac{1}{\rho}(p_\rho - \gamma p) + \rho^{-2}(p-1)^2(\alpha p + \beta p^{-1}) + \frac{\delta p(p+1)}{p-1}$	
$K_{VI} - model$	$p_{\rho\rho} = \varphi(\rho)p_\rho^2 - \xi(\rho)p_\rho + \rho^{-2}(\rho-1)^{-2}p(p-1)(p-\rho)[\alpha + \beta\rho p^{-2} + \zeta(\rho)]$	

Here $\varphi(z) = 0.5[p^{-1} + (p-1)^{-1} + (p-z)^{-1}]$, $\xi(z) = [z^{-1} + (z-1)^{-1} + (p-z)^{-1}]$, $\zeta(z) = [\gamma(z-1)(p-1)^{-2} + \delta z(z-1)(p-z)^{-2}]$. All above presented K_J - models admit exact solutions. Now let's present some of these solutions.

4.1.1 K_I - model

$$p_{\rho\rho} = 6p^2 + \rho. \quad (4.2)$$

4.1.2 K_{II} - model

$$p_{\rho\rho} = 2p^3 + \rho p + \alpha. \quad (4.3)$$

This models has the following particular solutions

$$p \equiv p(\rho; 1.5) = \psi - (2\psi^2 + \rho)^{-1}, \quad (4.4)$$

$$p \equiv p(\rho; 1) = -\frac{1}{\rho}, \quad (4.5)$$

$$p \equiv p(\rho; 2) = \frac{1}{\rho} - \frac{3\rho^2}{\rho^3 + 4}, \quad (4.6)$$

$$p \equiv p(\rho; 3) = \frac{3\rho^2}{\rho^3 + 4} - \frac{6\rho^2(\rho^3 + 10)}{\rho^6 + 20\rho^3 - 80}, \quad (4.7)$$

$$p \equiv p(\rho; 4) = -\frac{1}{\rho} + \frac{6\rho^2(\rho^3 + 10)}{\rho^6 + 20\rho^3 - 80} - \frac{9\rho^5(\rho^3 + 40)}{\rho^9 + 60\rho^6 + 11200}, \quad (4.8)$$

$$p \equiv p(\rho; 0.5\epsilon) = -\epsilon\psi \quad (4.9)$$

and so on. Here

$$\psi = (\ln \phi)_\rho, \quad \phi(\rho) = C_1 Ai(-2^{-1/3}\rho) + C_2 Bi(-2^{-1/3}\rho), \quad (4.10)$$

where $C_i = \text{consts}$ and $Ai(x), Bi(x)$ are Airy functions.

4.1.3 K_{III} - model

$$p_{\rho\rho} = \frac{1}{p}p_\rho^2 - \frac{1}{\rho}(p_\rho - \alpha p^2 - \beta) + \gamma p^3 + \frac{\delta}{p}. \quad (4.11)$$

This equation admits the infinite number exact solutions. For example, it has the following particular solutions (see e.g. [4])

$$p \equiv p(\rho; \mu, -\mu\kappa^2, \lambda, -\lambda\kappa^4) = \kappa, \quad (4.12)$$

$$p \equiv p(\rho; 0, -\mu, 0, \mu\kappa) = \kappa\rho, \quad (4.13)$$

$$p \equiv p(\rho; 2\kappa + 3, -2\kappa + 1, 1, -1) = \frac{\rho + \kappa}{\rho + \kappa + 1}, \quad (4.14)$$

$$p \equiv p(\rho; \mu, 0, 0, -\mu\kappa^3) = \kappa\sqrt[3]{\rho}, \quad (4.15)$$

$$p \equiv p(\rho; 0, -2\kappa, 0, 4\kappa\mu - \lambda^2) = \rho[\kappa(\ln \rho)^2 + \lambda \ln \rho + \mu], \quad (4.16)$$

$$p \equiv p(\rho; -\nu^2\lambda, 0, \nu^2(\lambda^2 - 4\kappa\mu), 0) = \frac{\rho^{\nu-1}}{\kappa\rho^{2\nu} + \lambda\rho^\nu + \mu}, \quad (4.17)$$

$$p \equiv p(\rho; 0.5\epsilon) = -\epsilon_1(\ln \phi)_\rho \quad (4.18)$$

and so on. Here

$$\phi(\rho) = \rho^\nu [C_1 J_\nu(\zeta) + C_2 Y_\nu(\zeta)], \quad (4.19)$$

where $C_i = \text{consts}$, $\zeta = \sqrt{\epsilon_1\epsilon_2}\rho$, $\nu = 0.5\alpha\epsilon_1$ and $J_\nu(\zeta), Y_\nu(\zeta)$ are Bessel functions.

4.1.4 K_{IV} - model

$$p_{\rho\rho} = \frac{1}{2p}p_\rho^2 + 1.5p^3 + 4\rho p^2 + 2(\rho^2 - \alpha)p + \frac{\delta}{p}. \quad (4.20)$$

This equation has the following particular solutions (see e.g. [4]):

$$p \equiv p(\rho; \pm 2, -2) = \pm \rho^{-1}, \quad (4.21)$$

$$p \equiv p(\rho; 0, -2) = -2\rho, \quad (4.22)$$

$$p \equiv p(\rho; 0, -3^{-2}2) = -3^{-1}2\rho, \quad (4.23)$$

$$p \equiv p(\rho; -m, -2(m-1)^2) = -[\ln H_{m-1}(\rho)]_\rho, \quad (4.24)$$

$$p \equiv p(\rho) = 2i\pi^{-0.5}e^{\rho^2}[iC + \text{erfc}(i\rho)]^{-1}, \quad (4.25)$$

$$p \equiv p(\rho) = 2\pi^{-0.5}e^{-\rho^2}[C - \text{erfc}(\rho)]^{-1}, \quad (4.26)$$

$$p \equiv p(\rho; 0.5\epsilon) = -\epsilon_1(\ln \phi)_\rho \quad (4.27)$$

and so on. Here

$$\phi(\rho) = [C_1 U(\zeta, 2^{0.5} \rho) + C_2 V(\zeta, 2^{0.5} \rho)] e^{0.5 \epsilon \rho^2}, \quad (4.28)$$

where $C_i = \text{consts}$, $\zeta = \alpha + 0.5\epsilon$, $\nu = 0.5\alpha\epsilon_1$ and U, V are parabolic cylinder functions, H_m are Hermite polynomials, erfc is the complementary error function (for detail see e.g. [4]).

4.1.5 K_V - model

$$p_{\rho\rho} = \left(\frac{1}{2p} + \frac{1}{p-1}\right)p_\rho^2 - \frac{1}{\rho}(p_\rho - \gamma p) + \rho^{-2}(p-1)^2(\alpha p + \beta p^{-1}) + \frac{\delta p(p+1)}{p-1}. \quad (4.29)$$

This equation admits the following particular solutions (see e.g. [4]):

$$p \equiv p(\rho; 0.5, -0.5\mu^2, \kappa(2-\mu), -0.5\kappa^2) = \kappa\rho + \mu, \quad (4.30)$$

$$p \equiv p(\rho; 0.5, \kappa^2\mu, 2\kappa\mu, \mu) = \kappa(\rho + \kappa)^{-1}, \quad (4.31)$$

$$p \equiv p(\rho; 0.125, -0.125, -\kappa\mu, \mu) = (\kappa + \rho)(\kappa - \rho)^{-1}, \quad (4.32)$$

$$p \equiv p(\rho; \mu, -0.125, -\mu\kappa^2, 0) = 1 + \kappa\rho^{0.5}, \quad (4.33)$$

$$p \equiv p(\rho; 0, 0, \mu, -0.5\mu^2) = \kappa e^{\mu\rho}, \quad (4.34)$$

$$p \equiv p(\rho) = -\epsilon_1 \rho (\ln \phi)_\rho \quad (4.35)$$

and so on. Here

$$\phi(\rho) = (\epsilon_2 \rho)^{-\nu} [C_1 M_{\kappa, \mu}(\epsilon_2 \rho) + C_2 W_{\kappa, \mu}(\epsilon_2 \rho)] e^{0.5 \epsilon_2 \rho}, \quad (4.36)$$

where $C_i = \text{consts}$ and M, W are Whittaker functions (for more exact details see e.g. [4]).

4.1.6 K_{VI} - model

$$p_{\rho\rho} = 0.5 \left(\frac{1}{p} + \frac{1}{p-1} + \frac{1}{p-\rho} \right) p_\rho^2 - \left(\frac{1}{\rho} + \frac{1}{\rho-1} + \frac{1}{p-\rho} \right) p_\rho + \rho^{-2}(\rho-1)^{-2} p(p-1)(p-\rho) [\alpha + \beta \rho p^{-2} + \gamma(\rho-1)(p-1)^{-2} + \delta \rho(\rho-1)(p-\rho)^{-2}]. \quad (4.37)$$

This equation is integrable and has the following particular solutions (see e.g. [4]):

$$p \equiv p(\rho; \mu, -\mu\kappa^2, 0.5, 0.5 - \mu(\kappa-1)^2) = \kappa\rho, \quad (4.38)$$

$$p \equiv p(\rho; 0, 0, 2, 0) = \kappa\rho^2, \quad (4.39)$$

$$p \equiv p(\rho; 0, 0, 0.5, -1.5) = \kappa\rho^{-1}, \quad (4.40)$$

$$p \equiv p(\rho; 0, 0, 2, -4) = \kappa\rho^{-2}, \quad (4.41)$$

$$p \equiv p(\rho; 0.5(\kappa + \mu)^2, -0.5, 0.5(\mu-1)^2, 0.5\kappa(2-\kappa)) = \rho(\kappa + \mu\rho)^{-1}, \quad (4.42)$$

$$p \equiv p(\rho; 0.5\kappa^2, -0.5\kappa^2, 0.5\mu^2, 0.5(1-\mu^2)) = \rho^{0.5} \quad (4.43)$$

and so on (for more exact details see e.g. [4]).

4.2 Nonintegrable 1.0 - models

1) Example 1:

$$p_{\rho\rho} = 0.5\Lambda - 1.5p_\rho^2. \quad (4.44)$$

2) Example 2:

$$p_{\rho\rho} = \xi(\rho)p + \frac{k}{p^3}, \quad (4.45)$$

3) Example 3:

$$p_{\rho\rho} = u(\rho)p + kp, \quad (4.46)$$

4) Example 4:

$$p_{\rho\rho} = \rho^{-1}(1-\rho)^{-1} \{[(\alpha + \beta + 1)\rho - \gamma]\dot{p} + \alpha\beta p\}. \quad (4.47)$$

5) Example 5:

$$p_{\rho\rho} = n(n-1) \sqrt[n]{A^2 p^{n-2}}. \quad (4.48)$$

This model has following solution

$$p = -A\rho^n. \quad (4.49)$$

5 D - models

In this section we would like to consider the so-called *D - models*, where $D = D(p, \rho)$ is an arbitrary function of ρ and p . Here some examples of such models.

5.1 Integrable D - models

1) D_I - models.

$D_{IA} - model$	$\ddot{D} = 6D^2 + t$
$D_{IB} - model$	$D_{aa} = 6D^2 + a$
$D_{IC} - model$	$D_{NN} = 6D^2 + N$
$D_{ID} - model$	$D_{HH} = 6D^2 + H$
$D_{IE} - model$	$D_{\rho\rho} = 6D^2 + \rho$
$D_{IF} - model$	$D_{pp} = 6D^2 + p$

(5.1)

2) D_{II} - models.

$D_{IIA} - model$	$\ddot{D} = 2D^3 + tD + \alpha$
$D_{IIB} - model$	$D_{aa} = 2D^3 + aD + \alpha$
$D_{IIC} - model$	$D_{NN} = 2D^3 + ND + \alpha$
$D_{IID} - model$	$D_{HH} = 2D^3 + HD + \alpha$
$D_{IIE} - model$	$D_{\rho\rho} = 2D^3 + \rho D + \alpha$
$D_{IIF} - model$	$D_{pp} = 2D^3 + pD + \alpha$

(5.2)

3) D_{III} - models.

$D_{IIIA} - model$	$\ddot{D} = D^{-1}\dot{D}^2 - t^{-1}(\dot{D} - \alpha D^2 - \beta) + \gamma D^3 + \delta D^{-1}$
$D_{IIIB} - model$	$D_{aa} = D^{-1}D_a^2 - a^{-1}(D_a - \alpha D^2 - \beta) + \gamma D^3 + \delta D^{-1}$
$D_{IIIC} - model$	$D_{NN} = D^{-1}D_N^2 - N^{-1}(D_N - \alpha D^2 - \beta) + \gamma D^3 + \delta D^{-1}$
$D_{IIID} - model$	$D_{HH} = D^{-1}D_H^2 - H^{-1}(D_H - \alpha D^2 - \beta) + \gamma D^3 + \delta D^{-1}$
$D_{IIIE} - model$	$D_{\rho\rho} = D^{-1}D_\rho^2 - \rho^{-1}(D_\rho - \alpha D^2 - \beta) + \gamma D^3 + \delta D^{-1}$
$D_{IIIF} - model$	$D_{pp} = D^{-1}D_p^2 - p^{-1}(D_p - \alpha D^2 - \beta) + \gamma D^3 + \delta D^{-1}$

(5.3)

4) D_{IV} - models.

$D_{IVA} - model$	$\ddot{D} = 0.5D^{-1}\dot{D}^2 + 1.5D^3 + 4tD^2 + 2(t^2 - \alpha)D + \beta D^{-1}$
$D_{IVB} - model$	$D_{aa} = 0.5D^{-1}D_a^2 + 1.5D^3 + 4aD^2 + 2(a^2 - \alpha)D + \beta D^{-1}$
$D_{IVC} - model$	$D_{NN} = 0.5D^{-1}D_N^2 + 1.5D^3 + 4ND^2 + 2(N^2 - \alpha)D + \beta D^{-1}$
$D_{IVD} - model$	$D_{HH} = 0.5D^{-1}D_H^2 + 1.5D^3 + 4HD^2 + 2(H^2 - \alpha)D + \beta D^{-1}$
$D_{IVE} - model$	$D_{\rho\rho} = 0.5D^{-1}D_\rho^2 + 1.5D^3 + 4\rho D^2 + 2(\rho^2 - \alpha)D + \beta D^{-1}$
$D_{IVF} - model$	$D_{pp} = 0.5D^{-1}D_p^2 + 1.5D^3 + 4pD^2 + 2(p^2 - \alpha)D + \beta D^{-1}$

(5.4)

5) D_V - models.

$D_{VA} - model$	$\ddot{D} = \phi\dot{D}^2 - t^{-1}(\dot{D} - \gamma D) + t^{-2}(D - 1)^2(\alpha D + \beta D^{-1}) + \delta\psi$
$D_{VB} - model$	$D_{aa} = \phi D_a^2 - a^{-1}(D_a - \gamma D) + a^{-2}(D - 1)^2(\alpha D + \beta D^{-1}) + \delta\psi$
$D_{VC} - model$	$D_{NN} = \phi D_N^2 - N^{-1}(D_N - \gamma D) + N^{-2}(D - 1)^2(\alpha D + \beta D^{-1}) + \delta\psi$
$D_{VD} - model$	$D_{HH} = \phi D_H^2 - H^{-1}(D_H - \gamma D) + H^{-2}(D - 1)^2(\alpha D + \beta D^{-1}) + \delta\psi$
$D_{VE} - model$	$D_{\rho\rho} = \phi D_\rho^2 - \rho^{-1}(D_\rho - \gamma D) + \rho^{-2}(D - 1)^2(\alpha D + \beta D^{-1}) + \delta\psi$
$D_{VF} - model$	$D_{pp} = \phi D_p^2 - p^{-1}(D_p - \gamma D) + p^{-2}(D - 1)^2(\alpha D + \beta D^{-1}) + \delta\psi$

(5.5)

Here $\phi = 0.5D^{-1} + (D - 1)^{-1}$, $\psi = D(D + 1)(D - 1)^{-1}$.

6) D_{VI} - models.

$D_{VIA} - model$	$\ddot{D} = \varphi(t)\dot{D}^2 - \xi(t)\dot{D} - \frac{D(D-1)(D-t)}{t^2(t-1)^2}[\alpha + \frac{\beta t}{D^2} + \zeta(t)]$
$D_{VIB} - model$	$D_{aa} = \varphi(a)D_a^2 - \xi(a)D_a - \frac{D(D-1)(D-a)}{a^2(a-1)^2}[\alpha + \frac{\beta a}{D^2} + \zeta(a)]$
$D_{VIC} - model$	$D_{NN} = \varphi(N)D_N^2 - \xi(N)D_N - \frac{D(D-1)(D-N)}{N^2(N-1)^2}[\alpha + \frac{\beta N}{D^2} + \zeta(N)]$
$D_{VID} - model$	$D_{HH} = \varphi(H)D_H^2 - \xi(H)D_H - \frac{D(D-1)(D-H)}{H^2(H-1)^2}[\alpha + \frac{\beta H}{D^2} + \zeta(H)]$
$D_{VIE} - model$	$D_{\rho\rho} = \varphi(\rho)D_\rho^2 - \xi(\rho)D_\rho - \frac{D(D-1)(D-\rho)}{\rho^2(\rho-1)^2}[\alpha + \frac{\beta \rho}{D^2} + \zeta(\rho)]$
$D_{VIF} - model$	$D_{pp} = \varphi(p)D_p^2 - \xi(p)D_p - \frac{D(D-1)(D-p)}{p^2(p-1)^2}[\alpha + \frac{\beta p}{D^2} + \zeta(p)]$

(5.6)

Here

$$\varphi(z) = 0.5[D^{-1} + (D - 1)^{-1} + (D - z)^{-1}], \quad (5.7)$$

$$\xi(z) = z^{-1} + (z - 1)^{-1} + (D - z)^{-1}, \quad (5.8)$$

$$\zeta(z) = \gamma(z - 1)(D - 1)^{-2} + \delta z(z - 1)(D - z)^{-2}. \quad (5.9)$$

5.2 Nonintegrable D - models

Nonintegrable D - models can be induced by some ODE's which are nonintegrable e.g. as in our previous papers [1]-[3].

5.3 Solutions

It is important that all above presented D - models admit exact solutions. Let's consider here one example. Let $D(p, \rho)$ has the form

$$D = 0.5(p + \rho). \quad (5.10)$$

Consider the D_{VID} - model [see the system (5.6)]. This model in fact has the form

$$p = -2\dot{H} - 3H^2, \quad (5.11)$$

$$\rho = 3H^2, \quad (5.12)$$

$$D_{HH} = \varphi(H)D_H^2 - \xi(H)D_H - \frac{D(D-1)(D-H)}{H^2(H-1)^2}[\alpha + \frac{\beta H}{D^2} + \zeta(H)] \quad (5.13)$$

$$\dot{\rho} = -3H(\rho + p), \quad (5.14)$$

Our aim is solve this system. For the particular case (5.10), the equation (5.13) has the following particular solution [4]

$$D = \kappa H^2, \quad (\kappa = const). \quad (5.15)$$

From (5.11)-(5.12) we get

$$\dot{H} = -\kappa H^2. \quad (5.16)$$

which has the solution

$$H = \frac{1}{\kappa(t - t_0)}. \quad (5.17)$$

6 Artificial two-dimensional models

FRW cosmological models that we considered above are one-dimensional. But for some reason we would like to have two-dimensional cosmological models in FRW spacetime. And there are (may be) no legal ways to construct two-dimensional models starting from one-dimensional models. If so let's try to use the "illegal" ways e.g. introducing the artificial "coordinate". As such artificial coordinate we can use e.g. one of physical parameters of the original model e.g. the cosmological constant (Λ). So we have may be two coordinates: one legal coordinate - t (time) and one "illegal" coordinate - Λ . Now we are ready to write our artificial two-dimensional cosmological models in FRW spacetime.

1) As an example, consider the following model e.g. for the scale factor [$a_\Lambda = da/d\Lambda$, $a_t = da/dt$ etc]:

$$a_t = 0.75(a^2)_\Lambda \quad (6.1)$$

or its twin

$$a_\Lambda = 0.75(a^2)_t \quad (6.2)$$

which are known to develop shocks [**Exercise 1:** *What means these shocks for the dynamics of the universe?*]. Eqs. (6.1) and (6.2) are nothing but the dispersionless Korteweg-de Vries equations (dKdVE) or Riemann equations. It is well-known that the equations (6.1) and (6.2) have the following solutions

$$a(\Lambda, t) = h(\Lambda - 0.75at) \quad (6.3)$$

and

$$a(t, \Lambda) = h(t - 0.75a\Lambda), \quad (6.4)$$

respectively. Here h is an arbitrary function. Also we see e.g. from the solution (6.3) that the velocity of a point of the wave, with constant amplitude a , is proportional to its amplitude leading to the "breaking" of the wave. Also we note that the wave also develops discontinuities in

its evolution [**Exercise 2:** *What means these discontinuities for the dynamics (evolution) of the universe?*]. Let's now we give the following particular solutions of the equations (6.1) and (6.2):

$$a = (\beta_1 + \beta_2 \Lambda)(\beta_3 - 1.5\beta_2 t)^{-1} \quad (6.5)$$

and

$$a = (\beta_1 + \beta_2 t)(\beta_3 - 1.5\beta_2 \Lambda)^{-1}, \quad (6.6)$$

respectively, where $\beta_i = \text{consts}$.

2) Our second example is the Euler-Tricomi equation

$$a_{tt} = ta_{\Lambda\Lambda}. \quad (6.7)$$

It is known that this equation has the following particular solution

$$a = \alpha(3\Lambda^2 + t^3) + \beta(\Lambda^3 + t^3\Lambda) + \delta(6t\Lambda^2 + t^4). \quad (6.8)$$

The twin of the equation (6.7) reads as

$$a_{\Lambda\Lambda} = \Lambda a_{tt}. \quad (6.9)$$

3) Let's now present the Dym equation

$$a_t = (a^{-0.5})_{\Lambda\Lambda\Lambda} \quad (6.10)$$

and its twin

$$a_{\Lambda} = (a^{-0.5})_{ttt}. \quad (6.11)$$

4) Our next example is given by [5]

$$(\ln a)_{tt} = a_{\Lambda\Lambda} \quad (6.12)$$

and its twin

$$(\ln a)_{\Lambda\Lambda} = a_{tt}. \quad (6.13)$$

5) Our last example is given by

$$a_t = \mu a_{\Lambda\Lambda}. \quad (6.14)$$

It is the heat equation and it is well-known that it has the following fundamental solution [as $a(\Lambda, t = 0) = \delta(\Lambda)$]

$$a = (4\pi\mu t)^{-0.5} e^{-0.25\mu^{-1}t^{-1}\Lambda^2}. \quad (6.15)$$

Note that the twin of the equation (6.14) looks like

$$a_{\Lambda} = \mu a_{tt}. \quad (6.16)$$

7 Conclusion

It is important to study the physical and mathematical nature of FRW cosmological models as they play a crucial role in modern cosmology. In this work, a new classes integrable and nonintegrable FRW cosmological models were proposed. To construct integrable models we implant integrable equations, in our Painlevé equations, into the body of the original gravitational equations. It is remarkable that all proposed integrable models admit exact solutions. For some of them, exact solutions are presented. Finally in Appendix, we extend the obtained results for g-essence models and its two reductions: k-essence and f-essence.

8 Appendix. Integrable and nonintegrable g-essence models and their k-essence and f-essence reductions

For g-essence we have

$$p = K, \quad \rho = 2XK_X + YK_Y - K, \quad (8.1)$$

where K is the Lagrangian for g-essence, X and Y are the kinetic terms for the scalar and spinor fields, respectively. Below we give some examples of integrable models for g-essence.

2) E_I - models.

$E_{IA} - model$	$\ddot{K} = 6K^2 + t$	(8.2)
$E_{IB} - model$	$K_{aa} = 6K^2 + a$	
$E_{IC} - model$	$K_{NN} = 6K^2 + N$	
$E_{ID} - model$	$K_{HH} = 6K^2 + H$	
$E_{IE} - model$	$K_{XX} = 6K^2 + X$	
$E_{IF} - model$	$K_{YY} = 6K^2 + Y$	

2) E_{II} - models.

$E_{IIA} - model$	$\ddot{K} = 2K^3 + tK + \alpha$	(8.3)
$E_{IIB} - model$	$K_{aa} = 2K^3 + aK + \alpha$	
$E_{IIC} - model$	$K_{NN} = 2K^3 + NK + \alpha$	
$E_{IID} - model$	$K_{HH} = 2K^3 + HK + \alpha$	
$E_{IIE} - model$	$K_{XX} = 2K^3 + XK + \alpha$	
$E_{IIF} - model$	$K_{YY} = 2K^3 + YK + \alpha$	

3) E_{III} - models.

$E_{IIIA} - model$	$\ddot{K} = K^{-1}\dot{K}^2 - t^{-1}(\dot{K} - \alpha K^2 - \beta) + \gamma K^3 + \delta K^{-1}$	(8.4)
$E_{IIIB} - model$	$K_{aa} = K^{-1}K_a^2 - a^{-1}(K_a - \alpha K^2 - \beta) + \gamma K^3 + \delta K^{-1}$	
$E_{IIIC} - model$	$K_{NN} = K^{-1}K_N^2 - N^{-1}(K_N - \alpha K^2 - \beta) + \gamma K^3 + \delta K^{-1}$	
$E_{IIID} - model$	$K_{HH} = K^{-1}K_H^2 - H^{-1}(K_H - \alpha K^2 - \beta) + \gamma K^3 + \delta K^{-1}$	
$E_{IIIE} - model$	$K_{XX} = K^{-1}K_X^2 - X^{-1}(K_X - \alpha K^2 - \beta) + \gamma K^3 + \delta K^{-1}$	
$E_{IIIF} - model$	$K_{YY} = K^{-1}K_Y^2 - Y^{-1}(K_Y - \alpha K^2 - \beta) + \gamma K^3 + \delta K^{-1}$	

4) E_{IV} - **models.**

$E_{IVA} - model$	$\ddot{K} = 0.5K^{-1}\dot{K}^2 + 1.5K^3 + 4tK^2 + 2(t^2 - \alpha)K + \beta K^{-1}$
$E_{IVB} - model$	$K_{aa} = 0.5K^{-1}K_a^2 + 1.5K^3 + 4aK^2 + 2(a^2 - \alpha)K + \beta K^{-1}$
$E_{IVC} - model$	$K_{NN} = 0.5K^{-1}K_N^2 + 1.5K^3 + 4NK^2 + 2(N^2 - \alpha)K + \beta K^{-1}$
$E_{IVD} - model$	$K_{HH} = 0.5K^{-1}K_H^2 + 1.5K^3 + 4HK^2 + 2(H^2 - \alpha)K + \beta K^{-1}$
$E_{IVE} - model$	$K_{XX} = 0.5K^{-1}K_X^2 + 1.5K^3 + 4XK^2 + 2(X^2 - \alpha)K + \beta K^{-1}$
$E_{IVD} - model$	$K_{YY} = 0.5K^{-1}K_Y^2 + 1.5K^3 + 4YK^2 + 2(Y^2 - \alpha)K + \beta K^{-1}$

(8.5)

5) E_V - **models.**

$E_{VA} - model$	$\ddot{K} = \phi\dot{K}^2 - t^{-1}(\dot{K} - \gamma K) + t^{-2}(K - 1)^2(\alpha K + \beta K^{-1}) + \delta\psi$
$E_{VB} - model$	$K_{aa} = \phi K_a^2 - a^{-1}(K_a - \gamma K) + a^{-2}(K - 1)^2(\alpha K + \beta K^{-1}) + \delta\psi$
$E_{VC} - model$	$K_{NN} = \phi K_N^2 - N^{-1}(K_N - \gamma K) + N^{-2}(K - 1)^2(\alpha K + \beta K^{-1}) + \delta\psi$
$E_{VD} - model$	$K_{HH} = \phi K_H^2 - H^{-1}(K_H - \gamma K) + H^{-2}(K - 1)^2(\alpha K + \beta K^{-1}) + \delta\psi$
$E_{VE} - model$	$K_{XX} = \phi K_X^2 - X^{-1}(K_X - \gamma K) + X^{-2}(K - 1)^2(\alpha K + \beta K^{-1}) + \delta\psi$
$E_{VD} - model$	$K_{YY} = \phi K_Y^2 - Y^{-1}(K_Y - \gamma K) + Y^{-2}(K - 1)^2(\alpha K + \beta K^{-1}) + \delta\psi$

(8.6)

Here $\phi = 0.5D^{-1} + (K - 1)^{-1}$, $\psi = K(K + 1)(K - 1)^{-1}$.

6) E_{VI} - **models.**

$E_{VIA} - model$	$\ddot{K} = \varphi(t)\dot{K}^2 - \xi(t)\dot{K} - \frac{K(K-1)(K-t)}{t^2(t-1)^2}[\alpha + \frac{\beta t}{K^2} + \zeta(t)]$
$E_{VIB} - model$	$K_{aa} = \varphi(a)K_a^2 - \xi(a)K_a - \frac{K(K-1)(K-a)}{a^2(a-1)^2}[\alpha + \frac{\beta a}{K^2} + \zeta(a)]$
$E_{VIC} - model$	$K_{NN} = \varphi(N)K_N^2 - \xi(N)K_N - \frac{K(K-1)(K-N)}{N^2(N-1)^2}[\alpha + \frac{\beta N}{K^2} + \zeta(N)]$
$E_{VID} - model$	$K_{HH} = \varphi(H)K_H^2 - \xi(H)K_H - \frac{K(K-1)(K-H)}{H^2(H-1)^2}[\alpha + \frac{\beta H}{K^2} + \zeta(H)]$
$E_{VIE} - model$	$K_{XX} = \varphi(X)K_X^2 - \xi(X)K_X - \frac{K(K-1)(K-X)}{X^2(X-1)^2}[\alpha + \frac{\beta X}{K^2} + \zeta(X)]$
$E_{VIF} - model$	$K_{YY} = \varphi(Y)K_Y^2 - \xi(Y)K_Y - \frac{K(K-1)(K-Y)}{Y^2(Y-1)^2}[\alpha + \frac{\beta Y}{K^2} + \zeta(Y)]$

(8.7)

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